- Two cochain map $f, g : C^\cdot \to D^\cdot$ are chain homotopic if $\exists\ h = \{h^k : C^k \to D^{k+1}\}$

s.t.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{k-1} \arrow[r] \arrow[dl, "h^{k-1}"'] & C^k \arrow[r, "d_k^C"] \arrow[d, "f-g"] \arrow[dl, "h^k"'] & C^{k+1} \arrow[r] \arrow[dl, "h^{k+1}"] & \cdots \\
\cdots \arrow[r] & D^{k-1} \arrow[r, "d_{k-1}^D"'] & D^k \arrow[r] & D^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

$$f^k - g^k = h^{k+1} \circ d_k^C - d_{k-1}^D h^k$$

At the first sight, it might look complicated, but it in fact has a strong geometric motivation.

<u>Recall</u> $\varphi, \psi : N \to M$ two continuous maps, they are <u>homotopic</u> if $\exists \{\varphi_t : N \to M\}_{t \in [0,1]}$ s.t. $\varphi_0 = \varphi$ and $\varphi_1 = \psi$.

One usually formulate in a different way: $\Phi : [0,1] \times N \to M$
$$(t, x) \to \varphi_t(x)$$

In particular, the map $\varphi_t : N \to M$ is the composition $\{t\} \times N \xrightarrow{i_t} [0,1] \times N \xrightarrow{\Phi} M$

and then $\varphi_t^* = (\Phi \circ i_t)^* = i_t^* \circ \Phi^*$.

$$\Rightarrow \quad \frac{\partial}{\partial t}\Big((\varphi_t^* \theta)(x)\Big) = \frac{\partial}{\partial t}\Big(\big((i_t^* \circ \Phi^*)(\theta)\big)(x)\Big) = \frac{\partial}{\partial t}\Big(\Phi^*(\theta)\ (t,x)\Big)$$

fixed $\theta \in \Omega^k(M)$

$x \in M$

$$= L_{\frac{\partial}{\partial t}}(\Phi^* \theta)$$

as a v.f. on $[0,1] \times N$

$$= d\, \iota_{\frac{\partial}{\partial t}}(\Phi^* \theta) + \iota_{\frac{\partial}{\partial t}} d(\Phi^* \theta)$$

Therefore

$$\int_0^1 \frac{\partial}{\partial t}(\varphi_t^* \theta)\, dt = \int_0^1 d\, \iota_{\frac{\partial}{\partial t}}(\Phi^* \theta)\, dt + \int_0^1 \iota_{\frac{\partial}{\partial t}} d(\Phi^* \theta)\, dt$$

$$= d \int_0^1 \iota_{\frac{\partial}{\partial t}}(\Phi^* \theta)\, dt + \int_0^1 \iota_{\frac{\partial}{\partial t}} \Phi^*(d\theta)\, dt$$

Define $h : \Omega^k(M) \longrightarrow \Omega^{k-1}(N)$ by $\int_0^1 \iota_{\frac{\partial}{\partial t}}(\Phi^*(-))\, dt$. Then

(c5) $\Rightarrow \quad \varphi_1^* \theta - \varphi_0^* \theta = d\, h(\theta) + h\, d(\theta)$. a htp relation

<u>Important</u>: if f, g are htp, then $f^* = g^* : H^*(C^\cdot; \mathbb{K}) \to H^*(D^\cdot; \mathbb{K})$.

e.g. For htp $\varphi_0, \varphi_1 : N \to M$, we have $(f_{\varphi_0})^* = (f_{\varphi_1})^* : H^*_{dR}(M;\mathbb{R}) \to H^*_{dR}(N;\mathbb{R})$.

using notation earlier

$\Rightarrow$ If N is htp equ to M, then $H^*_{dR}(M;\mathbb{R}) \simeq H^*_{dR}(M;\mathbb{R})$
(deformation retration)

e.g. $H^*_{dR}(\mathbb{R}^n;\mathbb{R}) = H^*_{dR}(\{pt\};\mathbb{R}) \overset{(c7)}{=\!=} \mathbb{R}$ only when $*=0$.

(c6)

<u>Exe</u>: Consider pair (M,A), and an ~~open~~ subset $U \subset M$ s.t. $\bar{U} \subset \mathrm{Int}(A)$, then the inclusion $(M \setminus U, A \setminus U) \to (M,A)$ induces an iso:

$$H^*_{dR}(M,A) \simeq H^*_{dR}(M \setminus U, A \setminus U).$$

<u>Def</u> (Eilenberg-Steenrod Axiom) A <u>cohomology theory</u> 上同调理论 consists of the following data for pair of space (M,A):

data
(i) $H^*(M,A;\mathbb{K})$ graded $\mathbb{K}$-module $\leftarrow$ $*$-th cohomology group of (M,A)

(ii) $f:(N,B)\to(M,A) \rightsquigarrow f^*: H^*(M,A;\mathbb{K})\to H^*(N,B;\mathbb{K})$
(s.t. $f(b)\in A\ \forall b\in B$)

(iii) $\delta^*: H^{*-1}(A;\mathbb{K})\to H^*(M,A;\mathbb{K})\quad \forall *$.

satisfying (C1) — (C7) axioms:

(C1) $\mathbb{1}:(M,A)\circlearrowleft \Rightarrow \mathbb{1}: H^*(M,A;\mathbb{R})\circlearrowleft$

(C2) $(g\circ f)^* = g^*\circ f^*$

(C3) $f\circ\delta = \delta\circ f|_A$

(C4) $\exists$ long exact seq: $\cdots\to H^{*-1}(A;\mathbb{K})\to H^*(M,A;\mathbb{K})\to H^*(M;\mathbb{K})\to H^*(A)\to\cdots$

(C5) $f\underset{htp}{\simeq} g \Rightarrow f^*=g^*$ on cohomology groups

(C6) For $\overline{U}\subset\mathrm{int}(A)$, inclusion $(M\backslash U, A\backslash U)\to(M,A)$ induces iso

(C7) $H^*(\{pt\};\mathbb{K})=\mathbb{K}$ only for $*=0$ (and 0 otherwise) $\leftarrow$ This makes cohomology theory non-trivial.

end of definition //

<u>Thm</u> $H^*_{dR}(M,A;\mathbb{R})$ form a cohomology theory.

- $H^*_{dR}(M;\mathbb{R})$ is called the absolute de Rham cohomology group.
- $H^*_{dR}(M,A;\mathbb{R})$ is called the relative de Rham cohomology group.

Rmk There are many cohomology theories: singular cohomology theory, cellur cohomology theory, sheaf cohomology theory...

Thm (Eilenberg-Steenrod) All cohomology theories are isomorphic.

In particular, $H^*_{dR}(M;\mathbb{R}) \simeq H^*_{sing}(M;\mathbb{R}) \simeq \cdots$ $\quad (=: H^*(M;\mathbb{R}))$

cohomology group of mfld M. over ($\mathbb{R}$)

$\Rightarrow$ Computing $H^*(M;\mathbb{R})$ from various approaches.

* Viewing $H^*_{dR}(M;\mathbb{R})$ from Eilenberg-Steenrod allows us to derive computational tools completely in an abstract level, without tracing back to the original def via forms.

Example (thm). Mayer – Vietoris seq from Eilenberg – Steenrod axioms:

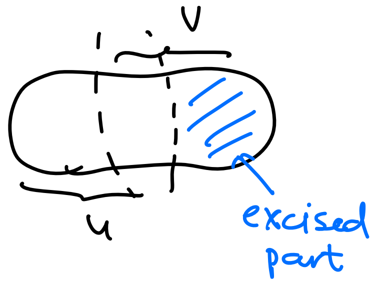
(1891 – 2002)

Assume mfd M is covered by open subset $\{U, V\}$. Then there are morphism $\delta^n : H^n(U\cap V; \mathbb{R}) \to H^{n+1}(M;\mathbb{R})$ s.t we have the following long exact seq

$$H^n(M;\mathbb{R}) \longrightarrow H^n(U;\mathbb{R}) \oplus H^n(V;\mathbb{R}) \longrightarrow H^n(U\cap V;\mathbb{R}) \xrightarrow{\delta^n} H^{n+1}(M;\mathbb{R}) \longrightarrow \cdots$$

Consider the following inclusions of pairs:

$$i : (\underset{\underset{M \setminus \text{blue part}}{\parallel}}{U}, \overset{\overset{V \setminus \text{blue part}}{\parallel}}{U\cap V}) \longrightarrow (\overset{\overset{U\cup V}{\parallel}}{M}, V)$$



by (c6) $\Longrightarrow$ $$H^*(U, U\cap V; \mathbb{R}) \simeq H^*(M, V; \mathbb{R})$$

Similarly $\quad j : (V, U\cap V) \longrightarrow (M, U)$

$$\overset{\text{by (c6)}}{\Longrightarrow} \quad H^*(V, U\cap V; \mathbb{R}) \longrightarrow H^*(M, U; \mathbb{R})$$

By (c4), the inclusion $U\cap V \hookrightarrow U$ induces a long exact seq ...
the inclusion $V \hookrightarrow M$ - - - - - - -

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(U, U\cap V; \mathbb{R}) \arrow[r] & H^*(U; \mathbb{R}) \arrow[r] \arrow[dr, phantom, "\circlearrowright"] & H^*(U\cap V; \mathbb{R}) \arrow[r] & H^{*+1}(U, U\cap V; \mathbb{R}) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^*(M, V; \mathbb{R}) \arrow[r] \arrow[u, "\simeq"'] & H^*(M; \mathbb{R}) \arrow[r] \arrow[u] & H^*(V; \mathbb{R}) \arrow[r] \arrow[u] & H^{*+1}(M, V; \mathbb{R}) \arrow[r] \arrow[u, "\simeq"'] & \cdots
\end{tikzcd}$$

This transfers to an elementary homological algebra problem

$$\begin{tikzcd}
\cdots \arrow[r] & X_1^n \arrow[r, "a^n"] & X_2^n \arrow[r, "c^n"] & X_3^n \arrow[r, "\delta_C^n"] & X_1^{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & Y_1^n \arrow[r, "b^n"'] \arrow[u, "f_1^n \text{ iso}"'] & Y_2^n \arrow[r, "d^n"'] \arrow[u, "f_2^n"'] & Y_3^n \arrow[r, "\delta_D^n"'] \arrow[u, "f_3^n"'] & Y_1^{n+1} \arrow[r] \arrow[u, "f_1^{n+1} \text{ iso}"'] & \cdots Y_2^{n+1}
\end{tikzcd}$$

a long exact seq $\Longrightarrow$

$$\cdots \to Y_2^n \xrightarrow{\alpha} X_2^n \oplus Y_3^n \xrightarrow{\beta} X_3^n \to Y_2^{n+1} \to \cdots$$

$\alpha = (f_2^n, d^n)$ $\beta = c^n - f_3^n$

This is the desired connecting morphism $\delta^n : H^*(U\cap V;\mathbb{R}) \to H^*(M;\mathbb{R})$

and $\delta^n := b^{n+1} \circ (f_1^{n+1})^{-1} \circ \delta_C^n$.

DIY: verify this is a long exact seq.

e.g. • $y \in Y_2^n \longmapsto (f_2^n(y), d^n(y)) \longmapsto c^n(f_2^n(y)) - f_3^n(d^n(y)) = 0$

$\Longrightarrow \operatorname{im}(\alpha) \subset \ker(\beta)$.

• $\forall\ (x,y) \in \ker\beta \quad (\Leftrightarrow c^n(x) - f_3^n(y) = 0)$.

Then $\delta_C^n(c^n(x)) = \delta_C^n(f_3^n(y)) = f_1^{n+1}(\delta_D^n(y)) = 0 \Rightarrow \delta_D^n(y) = 0$.

By exactness, $\exists\ y_* \in Y_2^n$ s.t. $d^n(y_*) = y$.

Meanwhile, compare $f_2^n(y_*)$ and x, we have $c^n(x) - c^n(f_2^n(y_*))$
$= c^n(x) - f_3^n(d^n(y_*)) = 0$.

$$\Rightarrow \quad x - f_2^n(y_*) \in \ker(c^n) = \mathrm{im}(a^n), \text{ so } \exists\, x_* \text{ s.t. } x - f_2^n(y_*) = a_n(x_*)$$

Therefore, consider $\tilde{y} = y_* + b^n \cdot (f_1^n)^{-1}(x_*)$, we have

$$\alpha(\tilde{y}) = \left(f_2^n(\tilde{y}),\ d^n(\tilde{y})\right)$$

$$= \left(f_2^n(y_*) + \left(f_2^n \circ b^n \circ (f_1^n)^{-1}\right)(x_*),\ \overset{y}{d^n(y_*)} + \overset{0}{\cancel{d^n \cdot b^n \cdot (f_1^n)^{-1}(x_*)}}\right)$$

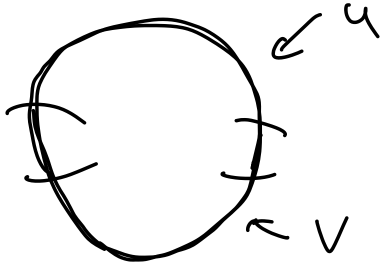
$$= \left(f_2^n(y_*) + \left(a_n \circ f_1^n \circ (f_1^n)^{-1}\right)(x_*),\ y\right) = \left(f_2^n(y_*) + x - f_2^n(y_*),\ y\right)$$

$$= (x, y) \quad \checkmark$$

2. Computational Examples

Recall $H^*_{dR}(\mathbb{R}^n;\mathbb{R}) = \begin{cases} \mathbb{R} & * = 0 \\ 0 & * \neq 0 \end{cases}$

e.g. $H^*_{dR}(S^1)$



$U \cap V$ = 2 copies of open interval $\simeq \mathbb{R} \sqcup \mathbb{R}$

$U \cup V = S^1$

(and $U, V \simeq \mathbb{R}$)

By MV sequence

$$0 \to H^0_{dR}(S^1;\mathbb{R}) \to H^0_{dR}(U;\mathbb{R}) \oplus H^0_{dR}(V;\mathbb{R}) \to H^0_{dR}(U\cap V;\mathbb{R}) \to H^1_{dR}(S^1;\mathbb{R}) \to H^1_{dR}(U;\mathbb{R}) \oplus H^1_{dR}(V;\mathbb{R}) \to \cdots$$

$$\iff \quad 0 \to \mathbb{R} \xrightarrow{\alpha} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\beta} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\delta} H^1_{dR}(S^1;\mathbb{R}) \to 0 \to \cdots$$

exactness of the sequence implies δ is surjective.

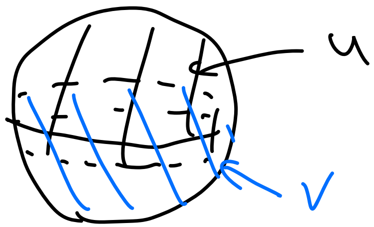
α: $1 \to (1,1)$

β: $(1,0) \to (1,1)$ and $(0,1) \to (-1,-1)$

δ: no need to know!

$$\dim H^1_{dR}(S^1;\mathbb{R}) = \dim(\operatorname{im}(\delta)) = 2 - \dim(\ker\delta)$$
$$= 2 - \dim(\operatorname{im}(\beta)) = 2-1 = 1$$

e.g. $H^*_{dR}(S^2; \mathbb{R})$



$$U \cap V \simeq S^1$$
$$U, V \simeq \{pt\}$$
$$U \cup V = S^2$$

By MV seq.

$$0 \to H^0_{dR}(S^2; \mathbb{R}) \to H^0_{dR}(U; \mathbb{R}) \oplus H^0_{dR}(V; \mathbb{R}) \to H^0_{dR}(U \cap V; \mathbb{R})$$
$$\to H^1_{dR}(S^2; \mathbb{R}) \to 0 \oplus 0 \to H^1_{dR}(U \cap V; \mathbb{R})$$
$$\to H^2_{dR}(S^2; \mathbb{R}) \to 0 \to \cdots$$

$$\Longleftrightarrow \quad 0 \to \mathbb{R} \xrightarrow{\alpha} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\beta} \mathbb{R}$$
$$\to H^1_{dR}(S^2; \mathbb{R}) \to 0 \to \mathbb{R}$$
$$\to H^2_{dR}(S^2; \mathbb{R}) \to 0 \cdots$$

$$\Longrightarrow \begin{cases} H^1_{dR}(S^2; \mathbb{R}) = 0 \\ H^2_{dR}(S^2; \mathbb{R}) = \mathbb{R} \end{cases}$$

In general, $H^*_{dR}(S^n; \mathbb{R}) = \begin{cases} \mathbb{R} & * = 0, n \\ 0 & * \neq 0, n \end{cases}$

$\Rightarrow$ ① All S^n $(n \geq 1)$ are orientable.

② S^n is not homotopy equ to S^m if $m \neq n$.

③ S^n $(n \geq 1)$ are not contractible

Note that ② $\Rightarrow$ $\mathbb{R}^n \not\simeq \mathbb{R}^m$ if $n \neq m$ (Note that simply from $H^*_{dR}(\mathbb{R}^n; \mathbb{R})$ one can't tell this).

Indeed, if $f: \mathbb{R}^n \simeq \mathbb{R}^m$, then

$$S^{n-1} \simeq \mathbb{R}^n \setminus \{0\} \simeq \mathbb{R}^m \setminus \{f(0)\} \simeq S^{m-1} \quad \checkmark$$

Rmk Here is a curious observation: one can take another open cover of S^2:

U_1 U_2 U_3

s.t. U_i, $U_i \cap U_j$, $U_1 \cap U_2 \cap U_3$ are all $\simeq$ open disk $\simeq \mathbb{R}^2$

Then run the MV-seq.

$$0 \to H^0_{dR}(S^2;\mathbb{R}) \to H^0_{dR}(U_1 \cup U_2;\mathbb{R}) \oplus \overset{\mathbb{R}}{\overset{\shortparallel}{H^0_{dR}(U_3;\mathbb{R})}} \to H^0_{dR}((U_1\cup U_2)\cap U_3;\mathbb{R})$$

$$\to H^1_{dR}(S^2;\mathbb{R}) \to H^1_{dR}(\cdot -) \to \cdots$$

$$\to H^2_{dR}(S^2;\mathbb{R}) \to \underbrace{H^2_{dR}(\cdot -) \to \;---}_{\text{some } \mathbb{R}\text{'s.}}$$

Moreover, by the exactness of LES from MV-seq, if $H^*_{dR}(U;\mathbb{R})$ and $H^*_{dR}(V;\mathbb{R})$ and $H^*_{dR}(U\cap V;\mathbb{R})$ are of finite-dim'l, then $H^*_{dR}(M;\mathbb{R})$ is of finite dim'l.

every mfd admits a good cover!

FACT: Every closed mfd M admits such a cover (called a finite good cover).

proved via Riem geo.

In particular, $\underline{H^*_{dR}(M;\mathbb{R}) \text{ is always of finite dim'l}}$.

deRham's Thm (for closed mfds)

e.g. compute $H^*_{dR}(\mathbb{C}P^n ; \mathbb{R})$ $(n \geq 2)$.

$$U = \mathbb{C}P^n \setminus \{[0, \cdots, 0, 1]\}$$

$$V = \mathbb{C}P^n \setminus \underset{\{[z_0,\cdots,z_n] \in \mathbb{C}P^n \mid z_n = 0\}}{\underset{\shortparallel}{\mathbb{C}P^{n-1}}} = \{[z_0, \cdots, z_n] \in \mathbb{C}P^n \mid z_n \neq 0\} \ \left(\underset{\text{dividing } z_n}{\simeq} \mathbb{C}^n \right)$$

$$U \cap V \simeq \mathbb{C}^n \setminus \{0\}$$

Observe:

$$\mathbb{C}P^{n-1} \xrightarrow{i} U \xrightarrow{\pi} \mathbb{C}P^{n-1}$$

$$[z_0, \cdots, z_{n-1}] \to [z_0, \cdots, z_{n-1}, 0]$$

$$\pi \circ i = \mathbb{1}_{\mathbb{C}P^{n-1}}$$

$$[z_0, \cdots, z_{n-1}, z_n] \to [z_0, \cdots, z_{n-1}]$$ ← This is well-defined b/c $[0, \cdots, 0, 1] \notin U$

$$U \xrightarrow{\pi} \mathbb{C}P^{n-1} \xrightarrow{i} U$$

$$[z_0, \cdots, z_{n-1}, z_n] \to [z_0, \cdots, z_{n-1}] \longrightarrow [z_0, \cdots, z_{n-1}, 0] \qquad i \circ \pi \underset{\text{htp}}{\simeq} \mathbb{1}_U.$$

where use htp $[0,1] \times U \to U$ by $(t, [z_0, \cdots, z_n]) \mapsto [z_0, \cdots, z_{n-1}, t z_n]$.

Passing to cohomology groups,

$$H^*_{dR}(U;\mathbb{R}) \simeq H^*_{dR}(\mathbb{CP}^{n-1};\mathbb{R})$$

$$H^*_{dR}(V;\mathbb{R}) \simeq H^*_{dR}(\mathbb{C}^n;\mathbb{R})$$

$$H^*_{dR}(U\cap V;\mathbb{R}) \simeq H^*_{dR}(\mathbb{C}^n\setminus\{0\};\mathbb{R}) \simeq H^*_{dR}(S^{2n-1};\mathbb{R}).$$

Apply MV-seq, we have

$$0 \to H^0_{dR}(\mathbb{CP}^n;\mathbb{R}) \to H^0_{dR}(\mathbb{CP}^{n-1};\mathbb{R}) \oplus \overset{\mathbb{R}}{\cancel{H^0_{dR}(\mathbb{C}^n;\mathbb{R})}} \to H^0_{dR}(S^{2n-1};\mathbb{R})$$

$$\to H^1_{dR}(\mathbb{CP}^n;\mathbb{R}) \to H^1_{dR}(\mathbb{CP}^{n-1};\mathbb{R}) \oplus \overset{0}{\cancel{H^1_{dR}(\mathbb{C}^n;\mathbb{R})}} \to H^1_{dR}(S^{2n-1};\mathbb{R}) \to \cdots$$

For simplicity, compute $n=2$, then

$$0 \to \mathbb{R} \to \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}$$

$$\to H^1_{dR}(\mathbb{CP}^2;\mathbb{R}) \to 0 \to 0$$

$$\to H^2_{dR}(\mathbb{CP}^2;\mathbb{R}) \to \mathbb{R} \to 0$$

$$\to H^3_{dR}(\mathbb{CP}^3;\mathbb{R}) \to 0 \to \mathbb{R}$$

$$\to H^4_{dR}(\mathbb{CP}^4;\mathbb{R}) \to 0 \to 0 \to 0 \cdots$$

$$\Rightarrow H^1_{dR}(\mathbb{C}P^2;\mathbb{R}) = H^3_{dR}(\mathbb{C}P^3;\mathbb{R}) = 0, \quad H^2_{dR}(\mathbb{C}P^2;\mathbb{R}) = H^4_{dR}(\mathbb{C}P^2;\mathbb{R}) = \mathbb{R}.$$

In general (by induction), we have

$$H^*_{dR}(\mathbb{C}P^n;\mathbb{R}) = \begin{cases} \mathbb{R} & * = 2k,\ k=0,\dots,n \\ 0 & \text{otherwise} \end{cases}$$

e.g. A useful formula to deal with product $M \times N$.

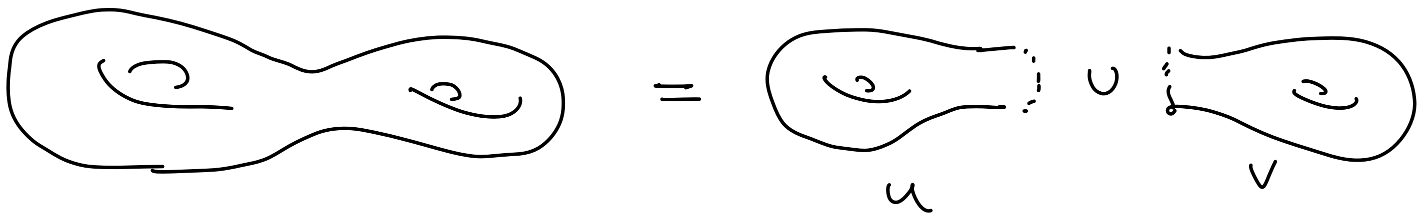
Künneth formula

(Exe) $$H^k_{dR}(M\times N;\mathbb{R}) \simeq \bigoplus_{\substack{p+q=k \\ 0\le p,q\le k}} H^p_{dR}(M;\mathbb{R}) \otimes_{\mathbb{R}} H^q_{dR}(N;\mathbb{R}).$$

$$\Rightarrow H^*_{dR}(T^2;\mathbb{R}) = H^*_{dR}(S^1\times S^1;\mathbb{R}) = \begin{cases} \mathbb{R}\otimes_{\mathbb{R}}\mathbb{R} = \mathbb{R} & * = 0 \\ \mathbb{R}\oplus\mathbb{R} & * = 1 \\ \mathbb{R}\otimes_{\mathbb{R}}\mathbb{R} = \mathbb{R} & * = 2 \end{cases}$$

In general, $H^k_{dR}(T^n;\mathbb{R}) = \mathbb{R}^{\binom{n}{k}}$ for $k=0,\dots,n$.

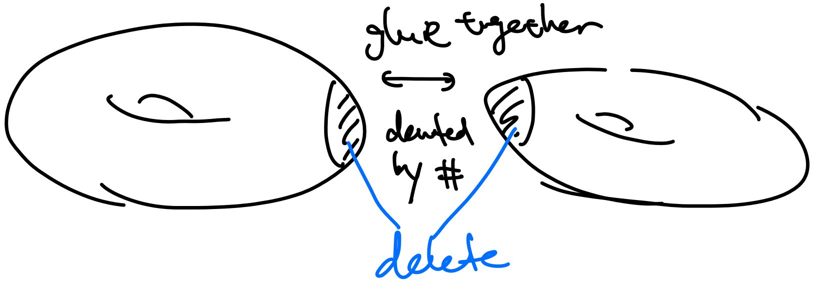
Rmk How about $\Sigma_{g\geq 2}$?



and $U \cap V =$  $(\simeq S^1)$

Then apply MV-seq.

Another way to calculate this is by "connected sum"



$$\overset{\text{DIY}}{\Longrightarrow} \quad H^1_{dR}(\overset{\Sigma_{g=2}}{\overset{\shortparallel}{\mathbb{T}^2 \# \mathbb{T}^2}};\mathbb{R}) \cong H^1_{dR}(\mathbb{T}^2;\mathbb{R}) \oplus H^1_{dR}(\mathbb{T}^2;\mathbb{R})$$

$$\left(\text{and } H^0_{dR}(\Sigma_{g=2};\mathbb{R}) = \overset{\cong\mathbb{R}}{H^2_{dR}(\Sigma_{g=2};\mathbb{R})}\right)$$

In general, when $\dim M = \dim N = n$,

$$H^k_{dR}(M \# N;\mathbb{R}) \cong H^k_{dR}(M;\mathbb{R}) \oplus H^k_{dR}(N;\mathbb{R}) \quad \text{for } 1 \leq k \leq n-1.$$

e.g. Aurcher product:

$$\alpha = [\theta_\alpha] \in H^k_{dR}(M;\mathbb{R}) \quad \text{and} \quad \beta = [\theta_\beta] \in H^l_{dR}(M;\mathbb{R})$$

$$\Rightarrow \quad \alpha \cup \beta := [\theta_\alpha \wedge \theta_\beta] \in H^{k+l}_{dR}(M;\mathbb{R})$$

Verification: • $d(\theta_\alpha \wedge \theta_\beta) = d\theta_\alpha \wedge \theta_\beta \pm \theta_\alpha \wedge d\theta_\beta = 0$ (b/c $d\theta_\alpha = d\theta_\beta = 0$)

• change θ_α to $\theta_\alpha + d\tau$ (still $\alpha = [\theta_\alpha + d\tau]$)

$$(\theta_\alpha + d\tau) \wedge \theta_\beta = \theta_\alpha \wedge \theta_\beta + d\tau \wedge \theta_\beta$$
$$= \theta_\alpha \wedge \theta_\beta + d(\tau \wedge \theta_\beta) \quad \text{b/c } d\theta_\beta = 0$$

$$\Rightarrow \quad [(\theta_\alpha + d\tau) \wedge \theta_\beta] = [\theta_\alpha \wedge \theta_\beta]. \qquad \checkmark$$

This is called the de Rham cup product of $H^*_{dR}(M;\mathbb{R})$.

$\Rightarrow$ $H^*_{dR}(M;\mathbb{R})$ admits a ring structure with unit (identity) equal to $\mathbb{1}$ = generator of $\mathbb{R} = H^0_{dR}(M;\mathbb{R})$. (= constant fcn 1).

$$\Rightarrow \quad H^*_{dR}(\mathbb{C}P^n;\mathbb{R}): \quad \mathbb{1}, \quad \underset{H^2_{dR}(\mathbb{C}P^n;\mathbb{R})}{\underset{\Uparrow}{\overset{\text{generator}}{c}}}, \quad \underset{H^4_{dR}(\mathbb{C}P^n;\mathbb{R})}{\underset{\Uparrow}{c \cup c}}, \quad \dots, \quad \underset{H^{2n}_{dR}(\mathbb{C}P^n;\mathbb{R})}{\underset{\Uparrow}{\overbrace{c \cup \cdots \cup c}^{n}}}$$